Superextension of Jordanian Deformation for $U(osp(1|2))$ and its Generalizations

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Abstract

We describe Jordanian "nonstandard" deformation of $U(osp(1|2))$ by employing the twist quantization technique. An extension of these results to $U(osp(1|4))$ describing deformed graded $D = 4$ AdS symmetries and to their super-Poincaré limit is outlined.

1 Introduction

It is well-known that the $U(sl(2))$ algebra with basic commutators

$$[h,e_{\pm}] = \pm e_{\pm}\,, \qquad [e_+,e_-] = 2h\,, \tag{1}$$

can be endowed with the following two inequivalent quantum deformations:

i) Drinfeld-Jimbo "standard" q-deformation with the following classical r-matrix ($q = 1-\gamma$)

$$r_{DJ} = \gamma e_+ \wedge e_-\,, \tag{2}$$

ii) Jordanian "nonstandard" quantum deformation, generated by the classical r-matrix

$$r_{_J} = \xi\, h \wedge e_+ \,. \tag{3}$$

The Hopf-algebraic structure of the Drinfeld-Jimbo deformation $U_q(sl(2))$ was given firstly in [1]–[3], and the Hopf algebra describing Jordanian deformation of $U(sl(2))$ was presented in [4].

The classical r-matrix (3) satisfies classical YB equation and its quantization can be described by so-called twist quantization method [5]. We recall that twist quantization of a Hopf algebra $H = (A, m, \Delta, S, \varepsilon)$ is given by the twisting two-tensor $F = \sum_i f_i^{(1)} \otimes f_i^{(2)}$ modifying the coproduct Δ and antipode S as follows:

$$\Delta \to \Delta_F = F\,\Delta F^{-1}\,, \tag{4a}$$

$$S \to S_F = u S u^{-1}, \qquad u = \sum_i f_i^{(1)}\, S\,(f_i^{(2)})\,. \tag{4b}$$

It should be stressed that the algebraic sector A of H remains unchanged.

The twisting two-tensor F for the Jordanian deformation of $U(sl(2))$ (see (3)) has been given firstly by Ogievetsky [6] in the following closed form

$$F_{_J} = \exp(\xi\, h \otimes E_+)\,, \tag{5}$$

where

$$E_+ = \frac{1}{\xi}\ln(1+\xi\, e_+) = e_+ + \mathscr{O}(\xi)\,. \tag{6}$$

The deformations (2) and (3) of $U(sl(2))$ provide important building blocks in the theory of quantum deformations of arbitrary Lie algebra. Similar role in the deformation theory of Lie superalgebras is played by the deformations of rank 1 superalgebra $osp(1|2)$, which is the supersymmetric extension of $sl(2) \simeq sp(2)$. Our first aim here is to generalize the Jordanian deformation of $U(sl(2))$ to the $U(osp(1|2))$ case. Further we present briefly the Jordanian deformation of $U(osp(1|4))$ as a special example of general framework presented by one of the authors in [7]. By interpreting $osp(1|4)$ as $D = 4$ AdS superalgebra we were able to obtain via contraction a new κ-deformation of $D = 4$ Poincaré superalgebra [8].

2 Jordanian Deformation of $U(osp(1|2))$

The classical r-matrices (2) and (3) are supersymmetrically extended as follows:

$$r^{susy}_{DJ} = \gamma(e_+ \wedge e_- + 2v_+ \wedge v_-)\,, \tag{7a}$$

$$r^{susy}_{J} = \xi\,(h \wedge e_+ - v_+ \wedge v_+)\,, \tag{7b}$$

where the odd generators $v_\pm$ of $osp(1,2)$ extend the $Sl(2)$ algebra (1) as follows:

$$[h, v_\pm] = \pm\frac{1}{2} v_\pm\,, \qquad \{v_+, v_-\} = -\frac{1}{2} h\,,$$
$$e_\pm = \pm 4\,(v_\pm)^2\,, \tag{8}$$

and in (7a–7b) for odd generators we define $a \wedge b = a \otimes b + b \otimes a$.

The quantization of the deformation (7a) is well known [9] as a particular case of the extension of Drinfeld-Jimbo quantization method to Lie superalgebras [10, 11]. The Jordanian quantization of $U(osp(1|2))$ generated by (7b) has been obtained quite recently, by the superextension of twist quantization procedure [12]. It should be mentioned that incomplete discussion of twist quantization of $U(osp(1|2))$ was presented earlier [13, 14], but explicite formulae for the twist tensor and all coproduct formulae have been given firstly in [12].

Let us recall that twisting element F should satisfy the cocycle equation

$$F^{12}(\Delta \otimes \mathrm{id})(F) = F^{23}(\mathrm{id} \otimes \Delta)(F)\,, \tag{9}$$

and the "unital" normalization condition

$$(\varepsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \varepsilon)(F) = 1\,. \tag{10}$$

We assume that the twisting two-tensor F_{SJ} describing the quantization of the classical r-matrix (7b) can be factorized as follows

$$F_{SJ} = F_S F_J\,, \tag{11}$$

where the "supersymmetric part" F_S depend on the odd generators $v_\pm$. Substituting (11) into (9) provides the following twisted cocycle condition for F_S

$$F^{12}_S(\Delta_J \otimes 1)(F_S) = F^{23}_S(1 \otimes \Delta_J)(F_S)\,, \tag{12}$$

where

$$\Delta_J(a) = F_J\,\Delta^{(0)}(a)\,F_J^{-1}\,, \tag{13}$$

and $\Delta^{(0)}(a) = a\otimes 1 + 1\otimes a$ for $a \in osp(1|2)$. Taking into consideration that the twist F_{SJ} for small values of ξ should have a form describing classical r-matrix (7b)

$$F_{SJ} = 1 + \xi(h\otimes e_+ - v_+\otimes v_+) + \mathscr{O}(\xi^2)\,, \tag{14}$$

one can write the solution of (12) in the following explicite form:

$$\begin{aligned} F_S &= 1 - 4\xi\,\frac{v_+}{e^\sigma + 1}\otimes\frac{v_+}{e^\sigma+1} \\ &= 1 - \xi\,\frac{v_+ e^{-\frac{1}{2}\sigma}}{\cosh\frac{1}{2}\sigma}\otimes\frac{v_+\,e^{-\frac{1}{2}\sigma}}{\cosh\frac{1}{2}\sigma}\,, \end{aligned} \tag{15}$$

where $\sigma = \frac{\xi}{2}E_+ = \frac{1}{2}\ln(1+\xi\,e_+)$ and $\Delta_J(\sigma) = \sigma\otimes 1 + 1\otimes\sigma$. One can show that

$$F_S^{-1} = \frac{\cosh\frac{1}{2}\sigma\otimes\cosh\frac{1}{2}\sigma + \xi\,v_+\,e^{-\frac{1}{2}\sigma}\otimes v_+\,e^{-\frac{1}{2}\sigma}}{\cosh\frac{1}{2}\Delta_J(\sigma)}\,. \tag{16}$$

Modifying (15) by a factor $\Phi = \Phi(\sigma)$ as follows

$$\widetilde{F}_S = \Phi F_S\,, \tag{17}$$

where

$$\Phi = \sqrt{\frac{(e^\sigma+1)\otimes(e^\sigma+1)}{2(e^\sigma\otimes e^\sigma+1)}}\,, \tag{18}$$

one obtains the unitary twist factor, i.e. $\widetilde{F}_S\widetilde{F}_S^* = \widetilde{F}_S^*\widetilde{F}_S = 1$, provided that the parameter ξ is purely imaginary. The choice (17) provides the following deformed coproducts of $U_\xi(osp(2|1))$

$$\widetilde{\Delta}_{SJ}(h) = h\otimes e^{-2\sigma} + 1\otimes h + \xi v_+ e^{-\sigma}\otimes v_+ e^{-2\sigma}\,, \tag{19a}$$

$$\widetilde{\Delta}_{SJ}(v_+) = v_+\otimes 1 + e^\sigma\otimes v_+\,, \tag{19b}$$

$$\begin{aligned} \widetilde{\Delta}_{SJ}(v_-) &= v_-\otimes e^{-\sigma} + 1\otimes v_- + \frac{\xi}{4}\Big\{\Big(\big\{h, e^\sigma\big\}\otimes v_+ e^{-2\sigma} \\ &\quad -\{h, v_+\}\otimes(e^\sigma - 1)e^{-2\sigma} \\ &\quad + 2v_+\otimes h - \Big\{h, \frac{v_+ e^\sigma}{e^\sigma+1}\Big\}\otimes(e^\sigma - 1)e^{-\sigma} \end{aligned}$$

$$+(e^{\sigma}-1)\otimes\left\{h,\frac{v_{+}}{e^{\sigma}+1}\right\}\Big),\frac{1}{e^{\sigma}\otimes e^{\sigma}+1}\Big\}, \tag{19c}$$

where

$$\Delta_{SJ}=\widetilde{F}_{S}F_{J}\Delta^{(0)}F_{J}^{-1}\widetilde{F}_{S}^{-1}. \tag{20}$$

Besides we get

$$\widetilde{S}_{SJ}(h) = -h e^{2\sigma}+\frac{1}{4}(e^{2\sigma}-1), \tag{21a}$$

$$\widetilde{S}_{SJ}(v_{+}) = -e^{-\sigma}v_{+}, \tag{21b}$$

$$\widetilde{S}_{SJ}(v_{-}) = -v_{-}e^{\sigma}+\xi h v_{+}e^{\sigma}-\frac{\xi}{4}v_{+}e^{\sigma}, \tag{21c}$$

where the formula (4b) with $F=\widetilde{F}_{S}F_{J}$ has been used.

Following the general framework of twist quantization the universal R-matrix is given by the formula ($F^{21}\equiv\sum_{i}f_{i}^{(2)}\otimes f_{i}^{(1)}$)

$$R=\widetilde{F}_{S}^{21}F_{J}^{21}F_{J}^{-1}\widetilde{F}_{S}^{-1}=\widetilde{F}_{S}^{21}R_{J}\widetilde{F}_{S}^{-1}, \tag{22}$$

where R_J is the Jordanian R-matrix describing the quantum algebra $U_{\xi}(sl(2))$:

$$R_{J}=F_{J}^{21}F_{J}^{-1}=e^{2\sigma\otimes h}e^{-2h\otimes\sigma}. \tag{23}$$

It can be added that

a) We discuss here the complex Lie superalgebra $osp(1|2)$. It appears that one can consider also the Jordanian deformation $U_{\xi}(osp(1|2;R)$ of the real form of $osp(1|2)$, in a way consistent with the real form of $sl(2)$ providing $sl(2;R)\simeq o(2,1)$ [12].

b) The real form of $U_{\xi}(osp(1|2))$ extending supersymetrically the algebra $U_{\xi}(o(2,1))$ describes deformed D=1 conformal superalgebra [15, 16]. It appears that possibly for the physical applications it is useful to use the new basis in $U(osp(1|2;R))$, with deformed $osp(1|2;R)$ superalgebra relations (see e.g. [17, 18]).

3 Beyond $osp(1|2)$

The next case of Jordanian deformation which is of physical interest in the $osp(1|2n)$ serie is n=2 [19], providing new quantum deformation of graded anti-de-Sitter al-

gebra [8]. In such a case using the Cartan-Weyl basis of $osp(1|4)$ (see also [7], where the notation is explained)

$$
\begin{aligned}
&(a)\; the\; rising\; generators: \;\; e_{1-2},\; e_{12},\; e_{11},\; e_{22},\; e_{01},\; e_{02}\;; \\
&(b)\; the\; lowering\; generators: \;\; e_{2-1},\; e_{-2-1},\; e_{-1-1},\; e_{-2-2},\; e_{-10},\; e_{-20}\;; \\
&(c)\; the\; Cartan\; generators: \;\; h_1 := e_{1-1},\; h_2 := e_{2-2}\;,
\end{aligned} \tag{24}
$$

one can write the following general r-matrix with its support in Borel sub-sueralgebra

$$
r(\xi_1,\xi_2) = r_1(\xi_1) + r_2(\xi_2)\;, \tag{25}
$$

$$
r_1(\xi_1) = \xi_1\left(\frac{1}{2}e_{1-1}\wedge e_{11} + e_{1-2}\wedge e_{12} - 2e_{01}\otimes e_{01}\right)\;, \tag{26}
$$

$$
r_2(\xi_2) = \xi_2\left(\frac{1}{2}e_{2-2}\wedge e_{22} - 2e_{02}\otimes e_{02}\right)\;. \tag{27}
$$

The twist quantization generated by the classical r-matrix (25) has the form [7, 19]

$$
F(\xi_1,\xi_2) = \widetilde{F}_2(\xi_2)\,F_1(\xi_1)\,, \tag{28}
$$

where F_1 is the twisting two-tensor corresponding to the classical r-matrix (26) and $\widetilde{F}_2$ is the two-twisting tensor corresponding to the r-matrix (27) with generators modified by suitable similarity map $\widetilde{e}_{ik} = \omega_{\xi_1}\, e_{ik}\, \omega_{\xi_1}^{-1}$ where

$$
\omega_{\xi_1} = \exp\left(\frac{\xi\,\sigma_{11}\,e_{1-2}\,e_{12}}{1-e^{2\sigma_{11}}}\right)\exp\left(\frac{1}{4}\,\sigma_{11}\right)\,, \tag{29}
$$

and $\sigma_{11} = \frac{1}{2}\ln(1+\xi_1 e_{11})$.

The 10 bosonic generators e_{mn} (see (24) if $m,n = \pm1,\pm2$ describe the AdS $O(3,2)$ generators, and the generators e_{0m} ($m = \pm1,\pm2$) define four odd supercharges. Introducing the AdS radius R and performing the limit $R\to\infty$ one can show [8] that the classical r-matrix (25) has the finite limit if $\xi = \xi_1 = \xi_2$ and ξ depends on R in the following way

$$
\xi(R) = \frac{i}{\kappa R}\,. \tag{30}
$$

In particular one obtains in the limit $R\to\infty$ from the classical r-matrix $r(\xi(R),\xi(R))$ (see (25)) the following super-Poincaré classical r-matrix:

$$
r_\kappa^{SUSY} = \frac{1}{\kappa}r^{LC} + \frac{2}{\kappa}\left(Q_1\wedge Q_1 + Q_2\wedge Q_2\right)\,, \tag{31}
$$

where $Q_m = \lim_{R\to\infty} (iR)^{-\frac{1}{2}} e_{0m}$ $(m = 1, 2)$ and r^{LC} describes the light-cone κ-deformation of Poincaré algebra [20, 21]. It apears that such a contraction limit $R \to \infty$ can be applied also to the twisted coproducts and twisted antipode of $U(osp(1|4))$ what provides new deformation of $D = 4$ Poincaré superalgebra.

In conclusion we would like to state that one can introduce by the contraction procedure two κ-deformations of $D = 4$, $N = 1$ supersymmetries

Drinfeld-Jimbo deformation $U_q(\mathfrak{osp}(1,4))$	$\xrightarrow[(q = \frac{1}{\kappa R};\quad R\to\infty)]{}$	Standard κ-deformed $D = 4$ Poincaré superalgebra [22]
Jordanian type deformation $U_{\xi_1,\xi_2}(\mathfrak{osp}(1\vert 4))$	$\xrightarrow[(\xi_1 = \xi_2 = \frac{i}{\kappa R};\ R\to\infty)]{}$	Light-cone κ-deformation of $D = 4$ Poincaré superalgebra

More detailed description of the second deformation is provided by the authors of the present report in [8].

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